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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 19 16997.12-M	2. GOVT ACCESSION NO. N/A AD-A095628	3. RECIPIENT'S CATALOG NUMBER N/A
4. TITLE (and Subtitle) 6 Numerical Calculation of Gravity-Capillary Interfacial Waves of Finite Amplitude		5. TYPE OF REPORT & PERIOD COVERED REPRINT
		6. PERFORMING ORG. REPORT NUMBER N/A
7. AUTHOR(s) 10 Jean-Marc Vanden-Broeck		8. CONTRACT OR GRANT NUMBER(s) 15 DAAG29-79-C-0222
9. PERFORMING ORGANIZATION NAME AND ADDRESS Stanford University Stanford, CA 94305 11 26 Feb 80		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS N/A
11. CONTROLLING OFFICE NAME AND ADDRESS US Army Research Office PO Box 12211 Research Triangle Park, NC 27709 12 5		12. REPORT DATE Sep 80
		13. NUMBER OF PAGES 4
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Submitted for announcement only

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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ARO 16997.12-M

Numerical calculation of gravity-capillary interfacial waves of finite amplitude

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Gravity-capillary progressive interfacial waves on the interface between two semi-infinite fluids of different densities are considered. An integro-differential equation for the unknown shape of the interface is derived. By introducing a mesh and finite difference, this equation is converted into a finite set of nonlinear algebraic equations. These equations are solved by Newton's method. In the limiting case of pure gravity waves, the results obtained are in good agreement with previous calculations. Two continuous families of capillary-gravity waves are studied. A generalization of Wilton's ripples for interfacial waves is presented.

I. INTRODUCTION

Over the past decade important progress had been achived in the calculation of free surface waves. Schwartz[1] extended Stokes' series for pure gravity waves to high order by computer and then recast these series as Padé approximants. High accuracy solutions were obtained in that way. Since then, this technique has been applied successfully to different kinds of surface waves such as solitary waves and gravity-capillary waves.[2-5]

On the other hand, a number of investigators[6-9] have obtained high accuracy solutions by using direct numerical approaches based on an integro-differential equation formulation. These calculations confirm the validity of the use of the Padé approximants as applied to surface waves. However, for very steep waves the numerical approach turns out to be more efficient than the Padé approximant technique.

Very little work has been done on the computation of nonlinear interfacial waves. The main results are contained in a paper by Holyer[10] who used Padé approximants to compute pure gravity interfacial waves.

In the present paper, we compute gravity-capillary interfacial waves by a direct numerical scheme. In Sec. II, the problem is formulated as a nonlinear integro-differential equation for the unknown shape of the interface. In Sec. III, this integral equation is solved by Newton's iterations. The mathematical formulation and the numerical method are similar in philosophy, if not in detail, to the procedure used by Schwartz and Vanden-Broeck[7] and Vanden-Broeck and Schwartz[8] to compute surface waves. The results are presented in Sec. IV. In the particular case of pure gravity waves our results agree with those obtained by Holyer.[10] Thus, the validity of the use of the Padé table as applied to interfacial waves is confirmed. Many different families of gravity-capillary interfacial waves exist. In Sec. IVB we shall study two of them. A direct generalization of Wilton's ripples is presented.

II. MATHEMATICAL FORMULATION

We consider two-dimensional progressive waves of wavelength λ and phase velocity c, propagating under the combined effect of gravity g and surface tension T along the interface between two fluids. We measure lengths in units of λ and velocities in units of c, so that all variables become dimensionless. We also choose a frame of reference in which the flows are steady.

The fluids are assumed to be incompressible and irrotational. Thus, we define stream functions ψ_1 and ψ_2 and potential functions ϕ_1 and ϕ_2 for the lower and upper fluids, respectively. Without loss of generality we choose $\psi_1=\psi_2=0$ on the interface and $\phi_1=\phi_2=0$ at one crest. Next, we introduce rectangular coordinates (x,y) with the x axis parallel to the velocities at infinite distance from the interface and with the y axis directed vertically upward. In addition, we define the capillary number κ, the wave speed parameter μ, and the density parameter ρ by the relations

$$\kappa=4\pi^2T/\rho_1 g\lambda^2 , \qquad (1)$$

$$\mu=2\pi c^2/g\lambda , \qquad (2)$$

$$\rho=\rho_2/\rho_1 . \qquad (3)$$

Here, ρ_1 and ρ_2 are, respectively, the densities in the lower and upper fluids.

On the interface, the Bernoulli equation and the pressure jump due to surface tension yield

$$\frac{\mu}{4\pi}(q_1^2-\rho q_2^2)+(1-\rho)y+\frac{\kappa}{4\pi^2}\frac{1}{R}=\frac{\mu}{4\pi}(1-\rho) , \qquad (4)$$

where q_1 and q_2 are the magnitudes of the velocity in the lower and upper fluids. R is the radius of curvature of the interface counted positive when the center of curvature lies on the side of the lower fluid. The choice of the constant in Eq. (4) fixes the origin of y as the undisturbed level for which the curvature is zero and $q_1=q_2=1$.

We shall consider $z=x+iy$ as an analytic function of the complex variables $f_1=\phi_1+i\psi_1$ and $f_2=\phi_2+i\psi_2$, respectively, on the half planes $\psi_1\le 0$ and $\psi_2\ge 0$. Then, the interface can be represented parametrically in two different ways, namely, $x(\phi_1,0_-)$, $y(\phi_1,0_-)$ and $x(\phi_2,0_+)$, $y(\phi_2,0_+)$, which we shall write $x_1(\phi_1)$, $y_1(\phi_1)$ and $x_2(\phi_2)$, $y_2(\phi_2)$.

To determine the shape of the interface we note that $\partial x/\partial\phi_1+i\partial y/\partial\phi_1-1$ and $\partial x/\partial\phi_2+i\partial y/\partial\phi_2-1$ vanish, respectively, at $\psi_1=-\infty$ and $\psi_2=+\infty$. Therefore, their real parts on the lines $\psi_1=0$ and $\psi_2=0$ are the Hilbert transforms of their imaginary parts:

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$$\frac{dx_1}{d\phi_1}=1-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{dy_1/d\phi_1'}{\phi_1'-\phi_1}\,d\phi_1' , \tag{5}$$

$$\frac{dx_2}{d\phi_2}=1+\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{dy_2/d\phi_2'}{\phi_2'-\phi_2}\,d\phi_2' . \tag{6}$$

We now use the assumed periodicity and symmetry of the surface to rewrite (5) and (6) in the forms

$$\frac{dx_1}{d\phi_1}=1-\int_0^{1/2}\frac{dy_1}{d\phi_1'}\,[\cot\pi(\phi_1'-\phi_1)-\cot\pi(\phi_1'+\phi_1)]d\phi_1' , \tag{7}$$

$$\frac{dx_2}{d\phi_2}=1+\int_0^{1/2}\frac{dy_2}{d\phi_2'}\,[\cot\pi(\phi_2'-\phi_2)-\cot\pi(\phi_2'+\phi_2)]d\phi_2' . \tag{8}$$

Since $\phi_1\neq\phi_2$ at two points in contact on the interface, we define the function

$$\phi_2=h(\phi_1) , \tag{9}$$

by the relations

$$x_1(\phi_1)=x_2[h(\phi_1)] , \tag{10}$$

$$y_1(\phi_1)=y_2[h(\phi_1)] , \tag{11}$$

$$h(0)=0 . \tag{12}$$

Thus, we have

$$\frac{dx_2}{d\phi_2}=\frac{dx_1}{d\phi_1}\left(\frac{dh}{d\phi_1}\right)^{-1} , \tag{13}$$

$$\frac{dy_2}{d\phi_2}=\frac{dy_1}{d\phi_1}\left(\frac{dh}{d\phi_1}\right)^{-1} . \tag{14}$$

Substituting Eqs. (13) and (14) into Eq. (8) we obtain

$$\frac{dx_1}{d\phi_1}\left(\frac{dh}{d\phi_1}\right)^{-1}=1+\int_0^{1/2}\frac{dy_1}{d\phi_1'}\{\cot\pi[h(\phi_1')-h(\phi_1)] - \cot\pi[h(\phi_1')+h(\phi_1)]\}d\phi_1' . \tag{15}$$

Finally, we rewrite condition (4) as

$$\frac{\mu}{4\pi}\left[1-\rho\left(\frac{dh}{d\phi_1}\right)^2\right]\left[\left(\frac{dx_1}{d\phi_1}\right)^2+\left(\frac{dy_1}{d\phi_1}\right)^2\right]^{-1}+(1-\rho)y(\phi_1) + \frac{\kappa}{4\pi^2}\left(\frac{dx_1}{d\phi_1}\frac{d^2y_1}{d\phi_1^2}-\frac{dy_1}{d\phi_1}\frac{d^2x_1}{d\phi_1^2}\right)\left[\left(\frac{dx_1}{d\phi_1}\right)^2+\left(\frac{dy_1}{d\phi_1}\right)^2\right]^{-3/2} = \frac{\mu}{4\pi}(1-\rho) . \tag{16}$$

In addition to the parameters κ, μ, and ρ, a given wave is characterized by a third parameter which is a measure of the wave amplitude. We choose this parameter to be the steepness s defined by

$$s=y(0)-y(\pi). \tag{17}$$

Dimensional analysis implies that a functional relationship should exist among the four numbers κ,μ,ρ, and s. We fix three of these parameters and we seek three functions

$$x_1(\phi_1),\ \ y_1(\phi_1),\ \ \text{and } h(\phi_1),\ \ \phi_1\in[0,1/2] ,$$

and a value for the fourth parameter that simultaneously satisfy Eqs. (7), (15), and (16). In the present paper, we fix κ,ρ, and s, and find μ as part of the solution.

III. NUMERICAL ANALYSIS

We seek a numerical solution of the integro-differential system of Eqs. (7), (15), and (16) by a finite difference method. Introducing a uniform mesh we have

$$\phi_1^i=(i-1)/2N,\ \ i=1,\ \ldots\ ,N+1. \tag{18}$$

Since the wave is symmetrical, we have

$$\left(\frac{dy_1}{d\phi_1}\right)_{\phi_1=\phi_1^1}=\left(\frac{dy_1}{d\phi_1}\right)_{\phi_1=\phi_1^{N+1}}=0 .$$

Thus, the unknown function $\partial y/\partial\phi$ can be represented by the vector of dimension $N-1$,

$$\mathbf{Y}'=(y_2',\ \ldots\ ,y_N') ,$$

where

$$y_i'=\left(\frac{dy_1}{d\phi_1}\right)_{\phi_1=\phi_1^i} .$$

We now define the midpoints $\gamma_i=\frac{1}{2}(\phi_i+\phi_{i+1})$, $i=1,\ldots,$ N and we represent the values of

$$\frac{dy_1}{d\phi_1},\ \frac{dx_1}{d\phi_1},\ y(\phi_1),\ h(\phi_1),\ \frac{dh}{d\phi_1},\ \frac{d^2x}{d\phi_1^2},\ \frac{d^2y}{d\phi_1^2} ,$$

at the points γ_i by the vectors $\mathbf{y}_m'$, $\mathbf{x}_m'$, $\mathbf{y}_m$, $\mathbf{h}_m$, $\mathbf{h}_m'$, $\mathbf{x}_m''$, and $\mathbf{y}_m''$.

We seek to satisfy the system (7), (15), and (16) at the points γ_i. The integrals in Eqs. (7) and (15) are evaluated at the points γ_i by the trapezoidal rule (which is of infinite order since the integrands are periodic). The integration is over the points ϕ_1^i. The singularities of the Cauchy principal values are automatically taken into account since the quadrature is symmetrical with respect to the singularities. Thus, we obtain from Eqs. (7) and (15),

$$\mathbf{x}_m'=1+A\mathbf{Y}' , \tag{19}$$

$$\mathbf{x}_m'=\mathbf{h}_m'+B(\mathbf{h})\mathbf{Y}' . \tag{20}$$

Here, A is a known matrix and B a matrix whose elements are nonlinear functions of the vector

$$\mathbf{h}=[h(\phi_1^2),\ \ldots\ ,h(\phi_1^N)] .$$

The vector $\mathbf{y}_m$ is expressed in terms of $\mathbf{Y}'$ by a sixth-order quadrature formula. Thus,

$$\mathbf{y}_m=Y_0+C\mathbf{Y}' , \tag{21}$$

where C is a known matrix. The elevation Y_0 of the interface at $\phi_1=0$ has to be found as part of the solution. Next, we express $\mathbf{x}_m''$, $\mathbf{y}_m''$, $\mathbf{h}$, $\mathbf{h}_m'$, and $\mathbf{y}_m'$ in terms of $\mathbf{x}_m'$, $\mathbf{Y}'$, and $\mathbf{h}_m$ by sixth-order interpolation and difference formulae.

Substituting Eqs. (19) and (21) into Eq. (15) at the points γ_i, $i=1,\ldots,N-1$ and Eqs. (16) at the points γ_i, $i=1,\ldots,N$ we obtain a system of $2N-1$ nonlinear algebraic equations for the $2N+1$ unknowns $\mathbf{Y}'$, $\mathbf{h}_m$, Y_0, and μ. Relations (12) and (17) provide two extra equations. Thus, for given values of κ,ρ, and s we have a system of $2N+1$ equations with $2N+1$ unknowns. This system is solved by Newton's iterations.

IV. DISCUSSION OF RESULTS

A. Pure gravity waves

Before proceeding to the general case where both gravity and surface tension are taken into account, we shall consider the limiting case of pure gravity waves.

In a recent paper, Holyer[10] computed accurate solutions for pure gravity interfacial waves up to a maximum value of the steepness at which the profile becomes vertical at some point. These solutions were obtained by calculating the coefficients in Stokes' expansion on a computer and than recasting the resulting high-order polynomials as Padé approximants.

The numerical procedure of Sec. III was used to compute gravity waves with $\rho = 0.1$. Preliminary computations showed that an accurate solution for $s > 0.1$ could not be computed with equal increments in the velocitv potential ϕ_1. The reason is that the spacing of the mesh points becomes too sparse near the crest where the velocity is small. A similar situation was previously encountered by Chen and Saffman,[6] Schwartz and Vanden-Broeck,[7] and Vanden-Broeck and Schwartz[8] in the case of surface waves. These authors found that this difficulty could easily be overcome by concentrating the mesh points near the crest by an appropriate change of variable. In the present problem, we introduce a new variable β by the relation

$$\beta = h(\phi_1) . \tag{22}$$

Here, the function h is defined by Eqs. (10)–(12). The numerical scheme was then used with equal increments in the new variable β. In Table I, we present values of μ for different values of the steepness s computed with $N = 15$, 20, and 25. The values for $N = 25$ have converged to five decimal places for $s < 0.13$ and to three decimal places for $s > 0.13$. These values agree with those presented graphically by Holyer.[10] Thus, the validity of the use of the Padé approximant method as applied to gravity interfacial waves is confirmed. A few typical profiles are shown in Fig. 1. For $s \sim 0.222$, the profile becomes vertical at a small distance from the crest. It is worthwhile mentioning that our numerical scheme appears to be more efficient than the Padé table method since the highest wave presented by Holyer corresponds to $s = 0.19$.

B. Gravity-capillary waves

Some insight into the problem can be gained by considering a solution in the form of a Stokes' expansion. Thus, we seek a solution as a Fourier expansion in the horizontal coordinate by assuming that the nth Fourier coefficient is nth order in the amplitude. In the first approximation, the waves are linear sine waves

TABLE I. Values of μ for $0.05 \leq s \leq 0.22$, $\kappa = 0$ and $\rho = 0.1$.

s \ μ	$N = 15$	$N = 20$	$N = 25$
0.05	0.835 097	0.835 105	0.835 107
0.095 493	0.880 605	0.880 620	0.880 624
0.127 324	0.930 479	0.930 483	0.930 484
0.159 155	0.996 561	0.996 498	0.996 480
0.190 986	1.062 550	1.062 075	1.061 947
0.206 90	1.137 163	1.135 659	1.135 199
0.21	1.149 422	1.147 491	1.146 868
0.22	1.196 938	1.192 921	1.190 878

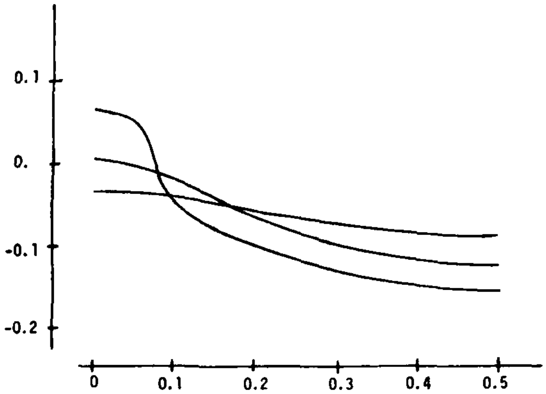

FIG. 1. Computed wave profiles for $\kappa = 0$, $\rho = 0.1$, and $s = 0.05$, 0.127, 0.2215.

and the dispersion relation is given by

$$\mu = \frac{1-\rho}{1+\rho} + \frac{\kappa}{1-\rho} . \tag{23}$$

It can easily be verified by substituting Eqs. (1) and (2) into Eq. (23) that linear waves of wavelength λ and λ/n travel with the same speed c if

$$\kappa = (1-\rho)/n , \tag{24}$$

where n is an integer greater than 1. This nonuniqueness in the first approximation implies that some of the coefficients in the Stokes' expansion will become infinite when κ assumes one of the values (24). In the particular case $\rho = 0$, these coefficients have been computed to fifth order by Wilton[11] and to 100th order by Hogan.[5] Solutions corresponding to the critical values (24) can be found by revoking Stokes' hypothesis and reordering the terms of the expansion. For example, Wilton[11] found two solutions for $\kappa = \frac{1}{2}$ and $\rho = 0$, i.e., for the critical value (24) corresponding to $n = 2$. The numerical work of Schwartz and Vanden-Broeck[7] shows clearly that these two solutions are, in fact, members

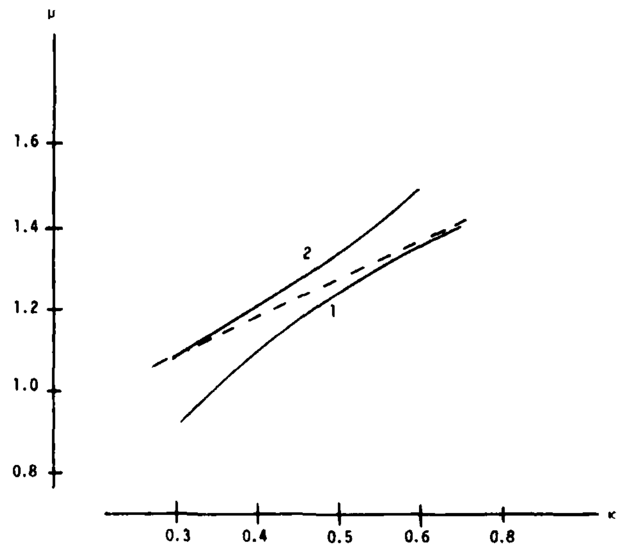

FIG. 2. Variation of speed parameter μ with capillary number for $s = 0.03$ and $\rho = 0.1$.

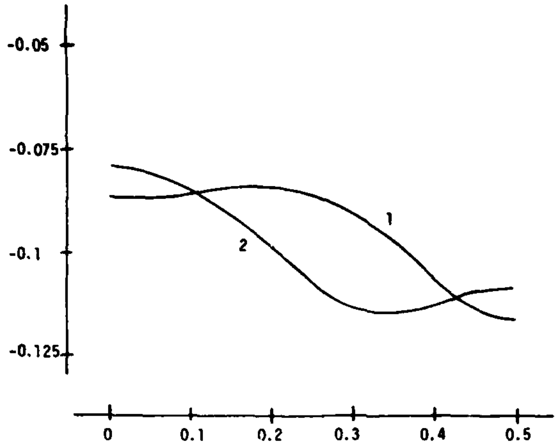

FIG. 3. Two wave profiles for $\kappa = 0.45$, $\rho = 0.1$, and $s = 0.03$.

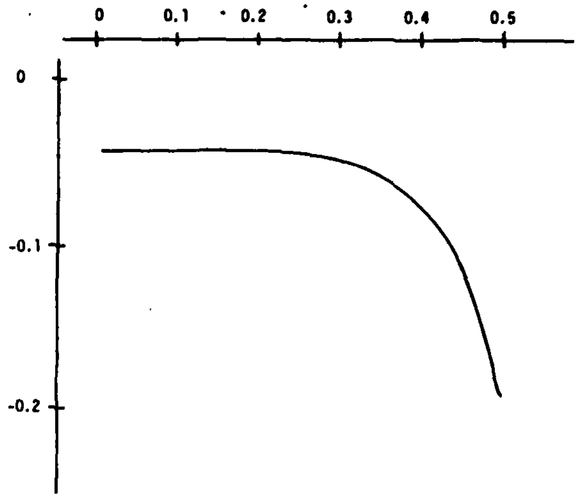

FIG. 4. Steep profile for $\kappa = 0.45$.

of two different families labeled 1 and 2. Families 1 and 2 agree, respectively, with the Stokes' expansion for $\kappa > \frac{1}{2}$ and $\frac{1}{3} < \kappa < \frac{1}{2}$.

In the present section, we use the numerical procedure of Sec. III, to compute the equivalent of these two families in the case $\rho = 0.1$. The critical value corresponding to $n = 2$ in Eq. (24) is then $\kappa = 0.45$. We started the iterations at $\kappa = 0.7$ and $s = 0.03$. A sine wave was used as an initial guess. The numerical scheme was found to converge to a wave of family 1. This result could be expected since family 1 agrees with the Stokes' expansion for $\kappa > 0.45$. This solution was used to compute the solution for a smaller value of κ and so on. Family 2 was computed in a similar way by starting the iterations at $\kappa = 0.4$ with a sine wave. The values of the speed parameter μ versus κ for $s = 0.03$ are presented in Fig. 2. Also shown is the infinitesimal wave solution (23). The two families could be extended to smaller and larger values of κ. This was not done since our purpose is simply to describe the behavior of the solutions near $\kappa = 0.45$. In Fig. 3, we present the profiles of the waves of families 1 and 2 for $\kappa = 0.45$ and $s = 0.03$. They are the direct extensions of the well-known Wilton ripples for interfacial waves with $\rho = 0.1$. In Fig. 4, we present a steep profile of a wave of family 1 for $\kappa = 0.45$ and $s = 0.15$. Although indiscernible in the figure the wave retains a slight crest dimple. Near the trough the slope of the profile varies rapidly and is equal to zero at $x = 0.5$. We were unable to compute waves of higher steepness for $\kappa = 0.45$. The numerical scheme is limited by the small values of the velocity at the trough in the upper fluid. We expect the limiting profile to exhibit a small trapped bubble at the trough with a stagnation point at the point of contact in the upper fluid.

ACKNOWLEDGMENTS

This work was supported by the Office of Naval Research, The National Science Foundation, the Air Force Office of Scientific Research, and the Army Research Office.

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